ELEMENTARY EXTENSIONS OF MODELS OF SET THEORY(1)

BY

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ABSTRACT

Model Theoretic methods are used to extend models of set theory while leaving specified sets fixed. In particular, every countable model $\mathfrak{A}$ of ZF has: (i) an extension leaving every set in $\mathfrak{A}$ fixed, and (ii) for each (in $\mathfrak{A}$) regular cardinal a an extension enlarging a but leaving each cardinal less than a fixed.

Let $\mathfrak{A} = \langle A, E \rangle$ be a model of Zermelo-Fraenkel set theory (ZF) and let a be a cardinal in $\mathfrak{A}$. We consider questions of the following kind:

I. Does $\mathfrak{A}$ have an elementary extension $\mathfrak{B} = \langle B, F \rangle$ such that a is enlarged but every cardinal $b < a$ in $\mathfrak{A}$ is left fixed?

II. Does $\mathfrak{A}$ have a proper elementary extension in which every cardinal of $\mathfrak{A}$ is left fixed?

We shall prove that the answer to I is "yes" if $\mathfrak{A}$ is countable and a is a regular cardinal in $\mathfrak{A}$ (Theorem 2.2). The answer to II is "yes" when $\mathfrak{A}$ is a countable model (Theorem 4.2). We shall also obtain a number of other results of this sort.

Section 1 contains the necessary notation. In Section 2 we consider questions like I. Section 3 contains theorems about extensions in which cardinals of the model have prescribed cofinalities as seen from the outside. Finally, Section 4 deals with questions like II. The Epilogue indicates how our theorems may be generalized to set theories other than ZF.

The results of [8], [17] yield (uncountable) models of ZF for which both I and II fail. By contrast, MacDowell and Specker [10] proved that the analogue of II is true for *every* model of number theory.

Section 1. We shall write $\mathfrak{A} \prec \mathfrak{B}$ if $\mathfrak{B}$ is an elementary extension of $\mathfrak{A}$; for this and other basic notions from model theory see Tarski and Vaught [13]. Two fundamental results about elementary extensions, proved in [13], are:

1.1. If $\mathfrak{A}_0 \prec \mathfrak{A}_1$ and $\mathfrak{A}_1 \prec \mathfrak{A}_2$ then $\mathfrak{A}_0 \prec \mathfrak{A}_2$.

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1.2. If $\mathfrak{A}_0, \mathfrak{A}_1, \cdots, \mathfrak{A}_\beta, \cdots$, $\beta < \alpha$ is an elementary chain (that is, $\gamma < \beta < \alpha$ implies $\mathfrak{A}_\gamma \prec \mathfrak{A}_\beta$), then the union $\bigcup_{\beta<\alpha} \mathfrak{A}_\beta$ is an elementary extension of each $\mathfrak{A}_\beta$.

By a theory T, we shall mean a set of sentences in a first order predicate logic. A sentence ϕ is said to be consistent with a theory T if $T \cup \{\phi\}$ is consistent. The notation $\mathfrak{A} \models \phi$ means that the sentence ϕ holds in the model $\mathfrak{A}$. We denote by $\sigma(v_0, \cdots, v_n)$ a formula having no free variable other than $v_0, \cdots, v_n$. A formula $\sigma(v_0)$ is consistent with T if $(\exists v_0)\sigma(v_0)$ is. If Σ is a set of formulas $\sigma(v_0)$, we shall say that $\mathfrak{A}$ *omits* Σ if there is no element $\alpha \in A$ which satisfies every $\sigma \in \Sigma$ in $\mathfrak{A}$. The *theory of* $\mathfrak{A}$, $Th(\mathfrak{A})$, is the set $\{\phi : \mathfrak{A} \models \phi\}$.

Our metalanguage will be an informal set theory which is like Zermelo-Fraenkel set theory (ZF) plus the axiom of choice. We shall be investigating models of the formal theory ZF for instance as formulated in [11]. Quotes around an informal statement will denote the corresponding formal statement of ZF.

Since we shall study ZF without the axiom of choice, we must be careful how we define certain standard notions of ZF. By a *cardinal* we shall mean an initial ordinal. The notation $|X| = \kappa$ means that the set X can be well ordered and has cardinal κ. If $\langle X, < \rangle$ is a simply ordered set, we say that Y is cofinal in $\langle X, < \rangle$ if $Y \subseteq X$ and for all $x \in X$ there is a $y \in Y$ with $x \leqq y$; the cofinality of $\langle X, < \rangle$ is the least ordinal β such that there is a set of order type β cofinal in $\langle X, < \rangle$. If α is a limit ordinal, then $cf(\alpha)$ denotes the cofinality of $\langle \alpha, < \rangle$. A *regular cardinal* is an infinite cardinal κ such that $\kappa = cf(\kappa)$. The result below can be proved in ZF (without the axiom of choice).

1.3. Let κ be an infinite limit ordinal. The following are equivalent:

(i) κ is a regular cardinal.

(ii) For every $\beta < \kappa$ and every function f on κ into β, there exists $\gamma < \beta$ such that $f^{-1}\{\gamma\}$ is cofinal in $\langle \kappa, < \rangle$.

(iii) For every $\beta < \kappa$ and every function f on κ into β, there exists $\gamma < \beta$ such that $|f^{-1}\{\gamma\}| = \kappa$.

One can also prove in ZF that:

1.4. For every infinite limit ordinal $\alpha, cf(\alpha)$ is a regular cardinal.

We shall use $\mathfrak{A} = \langle A, E \rangle, \mathfrak{B} = \langle B, F \rangle, \cdots$ to denote models of ZF, and we shall understand once and for all that A and E go with $\mathfrak{A}$, B and F go with $\mathfrak{B}$, etc. If $a \in A$, we shall write

$$\alpha_E = \{b \in A : bEa\}.$$

Thus a_E is the set of all "members" of a in the model $\mathfrak{A}$.

We must be careful to distinguish between the power of a in the model $\mathfrak{A}$, and the power of the set a_E. The former is an element b of $\mathfrak{A}$ such that $\mathfrak{A} \models b = |a|$, while the latter is the cardinal number $|a_E|$. We shall say that a is a cardinal of $\mathfrak{A}$ if $\mathfrak{A} \models$ "a is a cardinal".

Let $\mathfrak{A}$, $\mathfrak{B}$ be models of ZF and $\mathfrak{A} \prec_{e} \mathfrak{B}$. An element $a \in A$ is said to be *fixed* (by $\mathfrak{B}$) if $a_F = a_E$, and *enlarged* if $a_F \neq a_E$. Note that we always have $a_E \subseteq a_F$. If a is fixed then every cardinal less than a (in $\mathfrak{A}$) is fixed. Conversely, if $cf(a)$ and every cardinal less than a are fixed, a is fixed. In particular, if a is not regular and is enlarged then some cardinal less than a is enlarged.

Section 2. In this section we prove results concerning elementary extensions which leave one cardinal fixed and enlarge another cardinal. Our chief tool will be the following theorem for which a proof can be found in [3] or [16].

THEOREM 2.1. *Let T be a consistent theory in a countable logic and let S be a finite or countable set of sets of formulas $\sigma(v_0)$. Suppose that each $\Sigma \in S$ has the property*

> (*) *for each formula $\phi(v_0)$ which is consistent with T there exists $\sigma(v_0) \in \Sigma$ such that $\phi(v_0) \wedge \neg\sigma(v_0)$ is consistent with T.*

Then T has a countable model which omits each $\Sigma \in S$.

THEOREM 2.2. *Let $\mathfrak{A}$ be a countable model of* ZF *and let a be a regular cardinal of $\mathfrak{A}$. Then there is an elementary extension $\mathfrak{B} \succ \mathfrak{A}$ which leaves each $b \in a_E$ fixed but such that $|a_F| = \omega_1$.*

Proof. We shall prove that $\mathfrak{A}$ has a countable elementary extension $\mathfrak{A}^*$ which leaves each $b \in a_E$ fixed but enlarges a. After this has been shown, we may complete the proof in the following way. Form an elementary chain

$$\mathfrak{A} = \mathfrak{A}_0 \prec \mathfrak{A}_1 \prec \cdots \prec \mathfrak{A}_\alpha \prec \cdots \qquad \alpha < \mathfrak{A}_1,$$

where $\mathfrak{A}_{\alpha+1}$ is a model $\mathfrak{A}^*_\alpha$ enlarging α and leaving each $b \in a_E$ fixed, and for limit α, $\mathfrak{A}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{A}_\beta$. The required model $\mathfrak{B}$ is the union of the elementary chain,

$$\mathfrak{B} = \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha.$$

To construct the model $\mathfrak{A}^*$, we first enlarge our language by adding a new individual constant k_c for each $c \in A$, and one extra individual constant k. Let T be the theory with the following axioms.

$$Th(\langle A, E, c\rangle_{c\in A});$$

$$k \in k_\alpha;$$

$$k_b \neq k, \qquad \text{for each } b \in a_E.$$

T is clearly consistent. We note that a formula $\phi(v_0, k)$ is consistent with T if and only if

(1) $\langle A, E\ c\rangle_{c\in A} \models$ "the set $\{v_1 \in a : \exists v_0 \phi(v_0\ v_1)\}$ is cofinal with a".

For each $b \in a_E$, let Σ_b be the set of formulas

$$v_0 \neq k_c, \qquad c \in b_E;$$

$$v_0 \in k_b.$$

Let S be the countable set

$$S = \{\Sigma_b : b \in a_E\}.$$

Now consider any $\Sigma_b \in S$ and any formula $\phi(v_0, k)$ which is consistent with T. There exist e, $f \in A$ such that the following three statements are true in the model $\langle A, E, c \rangle_{c \in A}$:

$$\text{“}e = \{v_1 \in a : \exists v_0 \phi(v_0, v_1)\}\text{”};$$

$$\text{“}f \text{ is a function on } e \text{ into } b + 1\text{”};$$

$$\text{“for all } v_1 \in a, f(v_1) \text{ is the least ordinal } v_0 \text{ such that } v_0 = b \bigvee \phi(v_0, v_1)\text{”}.$$

Since a is a regular cardinal in $\mathfrak{A}$, and

$$\mathfrak{A} \models \text{“}e \text{ is cofinal with } a\text{”},$$

there exists $g \in A$ such that

$$\mathfrak{A} \models \text{“}g \leqq b \text{ and } f^{-1}(\{g\}) \text{ is cofinal with } a\text{”}.$$

If $g = b$ then the sentence

$$(\exists v_0)(\phi(v_0, k) \wedge \neg v_0 \in k_b)$$

is consistent with T. On the other hand, if $g \in b_E$ then

$$(\exists v_0)(\phi(v_0, k) \wedge v_0 = k_g)$$

is consistent with T. Hence Σ_b has the property (*). It follows from Theorem 2.1 that T has a countable model $\langle A^*, E^*, k_c, k \rangle_{c \in A}$ which omits each Σ_b, $b \in a_E$. Interpreting each k_c by c, $c \in A$, we see that $\mathfrak{A} \prec \mathfrak{A}^*$. Since each Σ_b is omitted, each $b \in a_E$ is fixed. But the interpretation of k belongs to $a_{E^*} - a_E$, so a is enlarged. Our proof is complete.

Using the two-cardinal theorem of Chang [1], we get at once the following corollary.

COROLLARY 2.3. (GCH). *Let $\mathfrak{A}$ be a countable model of* ZF, *let a be a regular cardinal of $\mathfrak{A}$, and let κ be a regular cardinal. Then there is an elementary extension $\mathfrak{B} \succ \mathfrak{A}$ such that $|b_F| = \kappa$ for all infinite $b \in a_E$, and $|a_F| = \kappa^+$.*

We now turn to some stronger theorems which are proved by the same methods as Theorem 2.2.

THEOREM 2.4. *In Theorem 2.2. the model $\mathfrak{B}$ may be chosen so that $\mathfrak{B}$ is not well-founded, and in fact so that the ordered set*

$$\langle a_F - a_E, F\rangle$$

contains a subset of order type $\eta \cdot \omega_1$ (η is the order type of the rationals).

Proof. We need only modify our construction of the model $\mathfrak{A}^*$ in such a way that $\langle a_{E^*} - a_E, E^*\rangle$ contains a subset of order type η. To do this we add to our language an individual constant k_c for each $c \in A$ and an individual constant k for each rational number r. Let T be the theory with the following axioms:

$$Th\ (\langle A, E, c\rangle_{c \in A});$$
$$k_r \in k_a, \quad r \text{ rational};$$
$$k_b \in k_r, \quad r \text{ rational and } b \in a_E;$$
$$k_r \in k_s, \quad r, s \text{ rational and } r < s.$$

We note that a formula

$$\phi(v_0, k_{r_1}, \cdots, k_{r_n}),$$

where $r_1 < \cdots < r_n$ are rational, is consistent with T if and only if $\langle A, E, c\rangle_{c \in A}$ satisfies the sentence

$$\text{"the set } \{v_1 \in a : (\exists v_0 v_2 \cdots v_n)\, \phi(v_0 v_1 v_2 \cdots v_n)$$
$$\text{and } v_1 \in v_2 \in \cdots \in v_n \in a\} \text{ is cofinal in } a\text{"}.$$

From here on we argue just as in the proof of Theorem 2.2.

A weaker form of the above theorem was announced in [8]. Using the methods of [8], Chang improved the result stated there. The present result gives still more information and the proof is different.

REMARK. In Theorem 2.4 the order type $\eta \cdot \omega_1$ is "best possible". To see this, note that an order type is embeddable in $\eta \cdot \omega_1$ if and only if every proper initial segment of it is finite or countable. Consider an arbitrary countable model $\mathfrak{A}$ of ZF and let a be a successor cardinal, $a = b^+$, in $\mathfrak{A}$. Let $\mathfrak{B} \succ \mathfrak{A}$ be an elementary extension which leaves b fixed. Then each $c \in a_F$ has power at most b in $\mathfrak{B}$, and hence $|c_F| \leqq \omega$. It follows that the order type of $\langle a_F, F\rangle$ has only finite or countable proper initial segments, and hence is embeddable in $\eta \cdot \omega_1$.

We now take up the problem of replacing the cardinal a of $\mathfrak{A}$ in Theorem 2.2 by a set of cardinals of $\mathfrak{A}$.

THEOREM 2.5. *Let $\mathfrak{A}$ be a countable model of* ZF, *and let $X, Y \subseteq A$ be non-empty sets of cardinals of $\mathfrak{A}$ such that X is infinite and*

$$X \subseteq \bigcap \{b_E : b \in Y\}.$$

Assume that:

(1) *for all* $a \in X$ *and* $b \in Y$ *there exists* $c \in Y$ *such that*

$$\mathfrak{A} \models \text{"for every function } f \text{ on } b \text{ into } a\text{, there exists } d \in a \text{ such that } |f^{-1}(\{d\})| \geqq c\text{"}.$$

Then:

(2) *there exists* $\mathfrak{B} \succ \mathfrak{A}$ *such that each* $a \in X$ *is fixed but for all* $b \in Y$, $|b_F| = \omega_1$.

Proof. We first prove:

(3) for each $b \in Y$ there is a countable $\mathfrak{B} \succ \mathfrak{A}$ such that each $a \in X$ is fixed but b is enlarged.

Let $b \in Y$ and let T be the theory with the axioms:

$$Th(\langle A, E, c\rangle_{c \in A});$$

$$k_a \in k, \text{ for each } a \in X;$$

$$k \in k_b;$$

$$k \neq k_c \quad \text{for each } c \in b_E;$$

$$\phi(k) \to |\{v_1 \in k_b : \phi(v_1)\}| > k_a,$$

for each sentence $\phi(k)$ and each $a \in \cap \{c_E : c \in Y\}$.

Since any finite number of axioms of T can be satisfied in $\mathfrak{A}$, T is consistent. A formula $\phi(v_0, k)$ is consistent with T if and only if for all $a \in \cap\{c_E : c \in Y\}$,

$$\mathfrak{A} \models \text{"}|\{v_1 \in b : \exists v_0 \phi(v_0, v_1)\}| > a\text{"},$$

and hence iff for some $c \in Y$,

$$\mathfrak{A} \models \text{"}|\{v_1 \in b : \exists v_0 \phi(v_0, v_1)\}| \geqq c\text{"}.$$

We let S be the set of all sets

$$\Sigma_a = \{v_0 \in k_a\} \cup \{v_0 \neq k_c : c \in a\},$$

where $a \in X$. Using (1) and our criterion for consistency, verify that each $\Sigma_a \in S$ satisfies (*). The proof of (3) is completed by applying Theorem 2.1 in the obvious way.

Observe that (1) holds for $\mathfrak{B}$ in place of $\mathfrak{A}$. We may thus prove (2) by using (3) ω_1 times being careful to enlarge each $b \in Y$ ω_1 times in the process.

We do not know whether condition (1) implies the following strong form of (2):

(4) There exists $\mathfrak{B} \succ \mathfrak{A}$ which leaves each $a \in X$ fixed but

$$|\bigcap\{b_F : b \in Y\}| = \omega_1.$$

Note that condition (1) is implied by the stronger condition

(1*) For every $b \in Y$ there exists $c \in Y$ such that in $\mathfrak{A}$, c is regular and $c \leqq b$.

One can prove that (1*) implies (2) simply by applying Theorem 2.1 ω_1 times, nstead of using the more complicated proof of Theorem 2.5. If every infinite sucessor cardinal of $\mathfrak{A}$ is regular, and Y is a set of infinite cardinals of $\mathfrak{A}$, then (1) and (1*) are almost the same and Theorem 2.5 is a corollary of Theorem 2.1. The special case of Theorem 2.5 where X is the set of all standard natural numbers is a trivial consequence of the compactness theorem.

Theorem 2.5 is of interest when X and Y are both sets of (non-standard) natural numbers of $\mathfrak{A}$. For in this case we have the following corollary.

COROLLARY 2.6. *Let $\mathfrak{A}$ be a countable model of* ZF *and let X be an infinite initial segment of the natural numbers of $\mathfrak{A}$. Then* (1′), (2′) *below are equivalent*:

(1′) *For all $a_1, a_2 \in X$, $a_1 a_2 \in X$.*

(2′) *There exists $\mathfrak{B} \succ \mathfrak{A}$ which leaves each $a \in X$ fixed but such that $|b_F| = \omega_1$ for each cardinal $b \notin X$ of $\mathfrak{B}$.*

REMARK. If in Theorem 2.5 we assume that $X \cup Y$ is the set of all cardinals of $\mathfrak{A}$, then conditions (1) and (2) are equivalent to each other.

Theorem 2.1 is false for uncountable theories (see [5]); thus our arguments in this section do not work for uncountable models of ZF. Indeed, it follows from the results of [8] that Theorem 2.2 fails for natural models $\langle R(\alpha), \varepsilon \rangle$ of ZF.

Section 3. In this section we apply the methods of Section 2 to uncountable models. We are not able to construct extensions which leave a cardinal a fixed, but we can control the cofinality of a as seen from the outside. We use a weak version of Theorem 2.1, due to Chang [2], which applies to uncountable languages.

THEOREM 3.1. *Let T be a consistent theory in a logic with at most κ symbols, where κ is a regular cardinal. Let S be a set of at most κ sets of formulas*

$$\Sigma = \{\sigma_\alpha(v_0) : \alpha < \kappa\},$$

where Σ has the property (*) *of* 2.1 *and also the property*:

(**) *For all* $\alpha < \beta < \kappa, \; T \vdash \forall v_0(\sigma_\beta(v_0) \rightarrow \sigma_\alpha(v_0)).$

Then T has a model of power κ which omits each $\Sigma \in S$.

In [2] the theorem was stated only for complete theories T, but Chang later pointed out that it holds for arbitrary consistent theories. The difficulty in extending the results of Section 2 to uncountable models is that the set of formulas

$$v_0 \neq k_b \wedge v_0 \in k_a, \ b \in a_E$$

does not satisfy the condition (**) when a_E is uncountable. However, the set of formulas

$$k_\alpha \in v_0 \wedge v_0 \in k_a, \ \alpha < \kappa$$

does satisfy (**) when the k_α represent increasing elements of a_E. In certain cases this will enable us to extend the model $\mathfrak{A}$ in such a way that a_E is cofinal in $\langle a_I, F\rangle$.

Our main theorem in this section is the following.

THEOREM 3.2. *Let $\mathfrak{A}$ be a model of ZF, let κ, λ be two regular cardinals at least one of which is $\geqq |A|$, and let a, b be distinct regular cardinals of $\mathfrak{A}$. Then there exists $\mathfrak{B} \succ \mathfrak{A}$ such that $\langle a_F, F\rangle$ has cofinality κ and $\langle b_F, F\rangle$ has cofinality λ.*

Actually we shall prove the following slightly more general result.

THEOREM 3.3. *Let $\mathfrak{A}$, κ, λ be as in Theorem 3.2, and let X, $Y \subseteq A$ be two disioint sets of regular cardinals of $\mathfrak{A}$. Then there exists $\mathfrak{B} \succ \mathfrak{A}$ such that:*

(i) *$\langle a_F, F\rangle$ has cofinality κ for all $a \in X$.*

(ii) *$\langle b_F, F\rangle$ has cofinality λ for all $b \in Y$.*

(iii) *The set of all ordinals of $\mathfrak{B}$ has cofinality κ.*

(iv) $|B| = \max(k, \lambda)$.

Proof. The argument is quite similar to the proof of Theorem 2.1, so we may omit some of the details. Let us first take up the case where $\kappa \geqq \lambda$. (The other case differs only for (iii).) Then $|A| \leqq \kappa$. By the compactness theorem, every model $\mathfrak{A}_0$ of ZF has an elementary extension $\mathfrak{A}_1$ such that

(1) $|A_1| = |A_0|$,

(2) $\mathfrak{A}_1$ contains an ordinal greater than every ordinal of $\mathfrak{A}_0$,

(3) for each regular cardinal a of $\mathfrak{A}_0$ there exists $c \in a_{E_1}$ which is greater than every element of a_{E_0}.

Here "greater" is with respect to the relation E_1. Using this observation κ times we may form an elementary chain

$$\mathfrak{A} = A_0 \prec \mathfrak{A}_1 \prec \cdots \prec \mathfrak{A}_\alpha \prec \cdots, \alpha < \kappa$$

whose union is a model $\mathfrak{B}_0$ of power κ such that:

(4) for all $a \in X$, $\langle a_{F0}, F_0\rangle$ has cofinality κ;

(5) $\langle \mathrm{Ord}, F_0\rangle$ has cofinality κ, where Ord is the set of all ordinals of $\mathfrak{P}_0$.

Enlarge the language by adding a new constant k_a for each $a \in B_0$. For each $a \in X$, let a_α, $\alpha < \kappa$, be a strictly increasing cofinal sequence in the ordered set $\langle a_{F_0}, F_0\rangle$. Let S contain, for each $a \in X$, the following set of formulas:

$$k_{a_\alpha} \in v_0 \wedge v_0 \in k_a, \qquad \alpha < \kappa.$$

Also, let c_α, $\alpha < \kappa$, be a strictly increasing cofinal sequence in $\langle \mathrm{Ord}, F_0 \rangle$, and let S contain the additional set of formulas:

$$k_{c_\alpha} \in v_0, \qquad \alpha < \kappa.$$

Now suppose $b \in Y$, add a new constant l to the language and let T_b be the theory with the following axioms:

$$Th(\langle B_0, F_0, k_c \rangle_{c \in B_0});$$

$$k_c \in l \wedge l \in k_b, \text{ for each } c \in b_{F_0}.$$

Using the assumptions that $X \cap Y = 0$ and X, Y contain only regular cardinals of $\mathfrak{B}_0$, it can be shown that T_b, S satisfy the hypotheses of Theorem 3.1. Therefore T_b has a model

$$\langle B', +', k_c, l \rangle_{c \in B_0}$$

of power κ which omits each $\Sigma \in S$. $\mathfrak{B}'$ is an elementary extension of $\mathfrak{B}_0$ such that for each $a \in X$ a_{F_0} is cofinal in $\langle a_{F'}, F' \rangle$, the set Ord is cofinal in the ordinals of $\mathfrak{B}'$, and the interpretation of l belongs to $b_F{}'$ but is greater than every element of b_{F_0}.

Well order the set Y, say $Y = \{b_\alpha, \alpha < \kappa\}$, and apply the above construction once for each $\alpha < \kappa$. This gives us an elementary chain

$$\mathfrak{B}_0 \prec \mathfrak{B}^1 \prec \cdots \prec \mathfrak{B}^\alpha \prec \cdots, \qquad \alpha < \kappa,$$

taking unions at the limit ordinals, where each $\mathfrak{B}^{\alpha+1}$ contains an element of b_α which is greater than every element of $(b_\alpha)_{F_0}$. Let $\mathfrak{B}_1$ be the union of the chain. Then $\mathfrak{B}_1$ has power κ and the properties

(6) for all $a \in X$, a_{F_0} is a cofinal in $\langle a_{F_1}, F_1 \rangle$;

(7) for all $b \in Y$, b_{F_1} contains an element greater than every element of b_{F_0};

(8) the ordinals of $\mathfrak{B}_0$ are cofinal in the ordinals of $\mathfrak{B}_1$.

Repeat the construction of $\mathfrak{B}_1$ from $\mathfrak{B}_0$ λ times. This yields an elementary chain

$$\mathfrak{B}_0 \prec \mathfrak{B}_1 \prec \cdots \prec \mathfrak{B}_\alpha \prec \cdots, \qquad \alpha < \lambda,$$

again taking unions at limit ordinals. The union

$$\mathfrak{B} = \bigcup_{\alpha < \lambda} \mathfrak{B}_\alpha$$

of this elementary chain has the desired properties (i)–(iv).

In case $\lambda > \kappa$ we argue in the same way except that κ and λ are interchanged and the set of formulas

$$k_{c_\alpha} \in v_0, \qquad \alpha < \lambda$$

is left out of S. Instead of the elementary chain

$$\mathfrak{B}_0 \prec \mathfrak{B}^1 \prec \cdots \prec \mathfrak{B}^\alpha \prec \cdots,$$

we form the chain

$$\mathfrak{B}_0 \prec \mathfrak{B}^{\,0} \prec \mathfrak{B}^1 \prec \cdots \prec \mathfrak{B}^\alpha \prec \cdots,$$

where $\mathfrak{B}^0$ contains an ordinal greater than all the ordinals of $\mathfrak{B}_0$. The theorem is proved.

The following questions are open. Can one prove a similar theorem for three regular cardinals κ, λ, μ and three sets X, Y, $Z \subseteq A$? Can the assumption $|A| \leqq \max(\kappa, \lambda)$ be dropped in Theorem 3.2?

Section 4. In this section we obtain some results concerning extensions of arbitrarily high power which leave particular sets fixed. It happens that the proofs are considerably simpler in the case of models in which the axiom of choice (AC) holds. We shall consider only such models in this section and indicate in an appendix how our proofs may be modified for the more general case. The methods used are related to those of [12]. In particular, we use a partition theorem of Erdös and Rado, [4].

For each cardinal κ and each ordinal α define inductively a cardinal 2^κ_α by $2^\kappa_0 = \kappa$ and for $\alpha > 0, 2^\kappa_\alpha = \bigcup\{2^\lambda | (\exists \beta < \alpha)(\lambda = 2^\kappa_\beta)\}$. Where 2^λ is cardinal exponentiation. In particular $2^\omega_\alpha = \beth_\alpha$ and thus the generalized continuum hypothesis may be stated as $\beth_\alpha = \omega_\alpha$ for all α. If $n \in \omega$ and A is a set let $A^{(n)}$ be the set of all subsets of A having exactly n elements.

THEOREM 4.1. (ERDÖS and RADO). *Suppose* $|A| > 2^\kappa_n$, κ *infinite,*

$$A^{(n+1)} = \bigcup \{C_i : i \in I\}$$

and $|I| \leqq \kappa$. *Then there is a* $B \subseteq A$ *and an* $i \in I$ *such that* $|B| > \kappa$ *and* $B^{(n+1)} \subseteq C_i$.

For a proof see [4] or the appendix to this paper where we state and prove a form of the theorem which does not depend on the axiom of choice.

THEOREM 4.2. *Suppose* $\mathfrak{A}$ *is a countable model of ZF and AC. Then for every linearly ordered set* $\langle X, < \rangle$ *there is a model* $\mathfrak{B} \succ \mathfrak{A}$ *such that*:

(i) *each* $a \in A$ *is left fixed*; *and*

(ii) $\langle X, < \rangle$ *is isomorphically embeddable in* $\langle B, F \rangle$ *(and hence* $|B| \geqq |X|$).

Proof. Consider $Th(\langle A, E, a \rangle_{a \in A})$. For each formula ϕ having $n+1$ free variables we add an n-ary function symbol f (the Skolem function) and the sentences:

(I) $$(\forall v_1 \cdots v_n)((\exists v_0)\phi(v_0, \cdots v_n) \rightarrow \phi(f(v_1, \cdots, v_n), v_1, \cdots v_n))$$

We also add a linearly ordered set $\langle X, < \rangle$ of new individual constants and the sentences:

(II) $x \in y$ (for all $x < y$ in X).

Let t be the set of terms formed from the Skolem functions and $\{(t_n, a_n)\}$ an enumeration of $t \times A$. Let $/$ be some object distinct from the elements of A and H a function: $\omega \to A \cup \{/\}$.

Corresponding to H we define the sets of sentences

(III) $\tau_n(x_1, \cdots x_k) = a$ (whenever τ_n has k free variables, $x_1 < \cdots < x_k$ in X and $H(n) = a$)

and

(IV) $\tau_n(x_1 \cdots x_k) \notin a_n$ (whenever τ_n has k free variables,

$x_1 < \cdots < x_k$ in X and $H(n) = / \cdot$)

Suppose I–IV were consistent with $Th(\langle A, E, a \rangle_{a \in A})$ and hence had a model. From I it follows that the closure of X under the Skolem functions is the universe of an elementary subsystem and from III and IV that any element in this closure is either already in A or is not in a_F for any a in A. Therefore to prove the theorem it will suffice to define H so that I–IV are consistent with $Th(\langle A, E, a \rangle_{a \in A})$.

The Skolem functions may not be definable in $Th(\langle A, E, a \rangle_{a \in A})$. But, since the axiom of choice holds in $\mathfrak{A}$, there is for every formula $\phi(v_0, \cdots v_k)$ and every $b \in A$ some function $f \in A$ defined on b^k which satisfies I. Indeed, if we thus limit their domain to a set of $\mathfrak{A}$ we can find functions corresponding to any finite number of Skolem terms. In particular, let $b \in A$ be a set of ordinals of $\mathfrak{A}$ and $(\tau_0(v_1, \cdots v_k), a_0)$ the first element in the enumeration of $t \times A$.

We define a partition on $b^{(k)}$ by letting $x_1 < \cdots < x_k$ be equivalent to $y_1 < \cdots < y_k$ if either (i) $\tau_0(x_1, \cdots, x_k) = \tau_0(y_1, \cdots y_k) \in a_0$ or (ii) neither $\tau_0(x_1 \cdots x_k)$ nor $\tau_0(y_1 \cdots y_k) \in a_0$. Since the axiom of choice holds in $\mathfrak{A}$ the theorem of Erdös and Rado holds in $\mathfrak{A}$. Therefore by assuming b to have large enough cardinality in $\mathfrak{A}$ we may find for each infinite cardinal in $\mathfrak{A}$ a set c in $\mathfrak{A}$ of that cardinality with $c^{(k)}$ entirely in one equivalence class. The equivalence classes correspond in an obvious way to $a_0 \cup \{//\}$ (where $//$ is some element of $\mathfrak{A}$ such that $// \notin a_0$). Thus there must be some $e_0 \in a_0 \cup \{//\}$ such that there exists $c \in A$ of arbitrarily large power in $\mathfrak{A}$ where $c^{(k)}$ lies entirely in the equivalence class corresponding to e_0.

Suppose $(\tau_1(v_1 \cdots v_l), a_1)$ is the next element in the enumeration of $t \times A$. By repeating the above argument there is an $e_1 \in a_1 \cup \{//\}$ and there are sets c of arbitrarily large cardinality in $\mathfrak{A}$ such that $c^{(k)}$, $c^{(l)}$ lie entirely in the equivalence classes corresponding to e_0, e_1. Proceed inductively to define $e_2, e_3, \cdots$. Using these to define the values of $H(0)$, $H(1), \cdots$ will make I–IV consistent since each finite subset of I-IV is now satisfiable in $\mathfrak{A}$. Theorem 4.2 is now proved.

In [17] there is an example of an uncountable model for which Theorem 4.2 fails. However we can prove some partial positive results about uncountable models.

THEOREM 4.3. *Suppose $\mathfrak{A}$ is a model of ZF and AC, $q_0, q_1, \cdots$ a denumerable sequence of ordinals of $\mathfrak{A}$, and*

$$U = \{a \in A : \text{For some } n, \ \mathfrak{A} \models a \in R(q_n)\}$$

Then for each linearly ordered set $\langle X, < \rangle$ there is a model $\mathfrak{A}$ with $U \subseteq B$ such that $\langle B, F, u\rangle_{u \in U}$ is elementarily equivalent to $\langle A, E, u\rangle_{u \in U}$, $u_E = u_F$ for every $u \in U$, and $\langle X, <\rangle$ is isomorphically embeddable in $\langle B, F\rangle$.

COROLLARY. *Theorem 4.2 holds for any model $\mathfrak{A}$ of ZF and AC in which there is a countable sequence cofinal in the ordinals of $\mathfrak{A}$.*

Proof. The difficulty in the proof is that the number of Skolem functions corresponding to $Th(\langle A, E, u\rangle_{u \in U})$ may be uncountable.

However there is an enumeration $\tau_0, \tau_1, \cdots$ of those Skolem terms which involve no constants corresponding to elements of U. Let t_{ij} be the set of Skolem terms formed by substituting constants corresponding to elements of $R(q_i)$ for some of the free variables of τ_j. For a fixed ij the elements of t_{ij} have bounded length. Therefore for every $b \in A$ there will be set (in $\mathfrak{A}$) of functions defined on b and corresponding to the Skolem terms of t_{ij}. Thus, we replace the denumerable sequence of Skolem terms by a denumerable sequence of sets of Skolem terms. The rest of the proof is essentially as in that of Theorem 4.2.

These methods may also be used to obtain "two-cardinal" theorems. We give two samples below. These are analogs of results in [12] and [15].

THEOREM 4.4. *Suppose $\mathfrak{A}$ is a countable model of ZF and AC, w and $b \in A$ and in $\mathfrak{A}$, $w = \omega$ and $b = \beth_{\omega_1}$. Then for every linearly ordered set $\langle X \ < \rangle$ there is a $\mathfrak{B} \succ \mathfrak{A}$ such that w is fixed but $\langle X, <\rangle$ is isomorphically embeddable in $\langle b_F, F\rangle$.*

Proof. We proceed parallel to the proof of Theorem 4.2. Consider $Th(\langle A, E, a\rangle_{a \in A})$. For each formula having $n + 1$ free variables we add an n-ary function symbol f and the axiom:

(I) $$(\forall v_1 \cdots v_n)((\exists v_0)\phi(v_0, \cdots v_n) \rightarrow \phi(f(v_1, \cdots, v_n), v_1, \cdots v_n))$$

Asbefore, we add a linearly ordered set $\langle X, <\rangle$of new constants, but now we add axioms saying they are in b:

(II) $$x \in y \in b \text{ (for all } x < y \text{ in } X).$$

Finally let H be a function: $\omega \to w_E \cup \{/\}$. We add axioms for each Skolem term τ_n:

$$\text{(III)} \qquad \tau_n(x_1 \cdots x_k) = a \ \ (\text{if } x_1 < \cdots < x_k \text{ and } H(n) = a \in w_E)$$

and

$$\text{(IV)} \qquad \tau_n(x_1 \cdots x_k) \notin w \ \ (\text{if } x_1 < \cdots < x_k) \text{ and } H(n) = /.)$$

As in the proof of Theorem 4.3, if we define some H such as to make I–IV consistent the theorem will be established. The definition of H is done inductively as before with one significant difference. Instead of considering arbitrary sets of ordinals, we now consider only sets $c \subseteq b$.

Further, instead of seeking an equivalence class with sets c of "arbitrarily large" cardinality in $\mathfrak{A}$ with $c^{(k)}$ in that class, we now seek an equivalence class such that for each $\alpha < \omega_1$ there is a set c of power $\beth_\alpha$ (in $\mathfrak{A}$) with $c^{(k)}$ in that class. Since there are only a countable (in $\mathfrak{A}$) number of equivalence classes we can always find one such. The proof otherwise is as in that of Theorem 4.2.

THEOREM 4.5. *Suppose $\mathfrak{A}$ is a countable model of ZF and AC, $w, b \in A$ and in $\mathfrak{A}$, $w = \omega$ and $b = \beth_\omega$. Then for every linearly ordered set $\langle X, < \rangle$ there is a model $\mathfrak{B} \succ \mathfrak{A}$ such that w_F is countable and $\langle X, < \rangle$ is isomorphically embeddable in $\langle b_F, F \rangle$.*

Proof. The proof is similar to that of the previous theorem except now $H: \omega \to \{0, 1\}$ and the axioms III and IV are:

$$\text{(III)} \quad \tau_n(x_1, \cdots, x_k) = \tau_n(y_1, \cdots, y_k) \ (\text{whenever } x_1 < \cdots < x_k \text{ and } y_1 < \cdots < y_k \text{ and } H(n) = 1)$$

$$\text{(IV)} \qquad \tau_n(x_1 \cdots x_k) \notin w \ \ (\text{whenever } x_1 < \cdots < x_k \text{ and } H(n) = 0).$$

Notice that Theorem 4.4 is still true if we assume that in $\mathfrak{A}$, w is a regular cardinal and $b = 2^w_{w^+}$. Moreover, Theorem 4.5 is still true if we assume that in $\mathfrak{A}$, w is an infinite cardinal and $b = 2^w_\omega$. In each case, the original proofs go through with the obvious modifications. Combining Theorem 4.5 with the two-cardinal theorem of Vaught [15], we obtain a corollary.

COROLLARY. *Let $\mathfrak{A}$ be a countable model of ZF and AC, $a, b \in A$, and in $\mathfrak{A}$, a is a cardinal and $b = 2^a_\omega$. Then for any two infinite cardinals $\kappa \leqq \lambda$, there exists $\mathfrak{B} \succ \mathfrak{A}$ such that $|a_F| = \kappa$, $|b_F| = |B| = \lambda$.*

Appendix to Section 4. Throughout this paper we have assumed the Axiom of Choice in the meta-language. In Section 4, moreover, we restricted ourselves to models of ZF + AC. It is the purpose of the appendix to indicate how this latter restriction may be removed. Notice first that since the definition of $\beth_\alpha$

implicitly assumes the Axiom of Choice, Theorems 4.4 and 4.5 may not even be meaningful for models in which the Axiom of Choice does not hold. However, we shall give a new definition of $\beth_\alpha$ below which does not assume the Axiom of Choice. With this definition, we have:

THEOREM. *Theorems 4.2, 4.3, and 4.5 are true for models of ZF in general. Theorem 4.4 is true for any model of ZF in which ω_1 is regular.*

In Section 4 the fact that we were dealing with models of ZF + AC was used in two ways: first, to find functions in the model which act as "partial" Skolem functions, and second, to guarantee that the Erdös-Rado Theorem was true in the model.

The first of these uses can be eliminated as follows. Suppose $\mathfrak{A}$ is a model of ZF, $b \in \mathfrak{A}$, $\phi(v_0, \cdots, v_n)$ a formula; then there is a function $f \in A$ with domain b^n such that:

$$v_1 \in b \wedge \cdots \wedge v_n \in b \wedge (\exists v_0)\phi(v_0, v_1, \cdots, v_n) \rightarrow$$
$$(\exists v_0)(v_0 \in f(v_1, \cdots, v_n) \wedge \phi(v_0, \cdots, v_n)).$$

Where in Section 4 the Skolem function determined a partition indexed over the elements of a set, we now have one indexed over the subsets of that set. However this leads to only minor modifications in the proofs. The situation is even simpler in the proofs of Theorems 4.4 and 4.5 since the set in question is ω which is well-ordered without the Axiom of Choice.

The rest of this appendix is devoted to the second problem: that of restating and proving the theorem or Erdös and Rado (Theorem 4.1) in a form which does not depend on the Axiom of Choice. This will involve some *ad hoc* definitions of cardinal arithmetic.

We define a cardinal as an initial ordinal, κ^+ is the least cardinal $> \kappa$. The cardinality, $|X|$, of a set X is the supremum of those cardinals which can be mapped one-one into X. It is conceivable that some infinite cardinal κ^+ is the union of κ sets each of power κ. However, it can be shown that κ^{++} is not the union of κ sets of power κ. Thus we can define for infinite cardinals κ, κ^0 = largest cardinal which is the union of κ sets each of power κ. The Axiom of Choice implies $\kappa^0 = \kappa$. For each set x and eachordinal α we define a cardinal $2(x, \alpha)$ inductively by:

$$2(x,0) = |x|;$$

$$2(x, \alpha+1) = \text{least cardinal} > 2(x,\alpha)^0 \text{ and}$$
$$\geqq |\text{power set of } 2(x,\alpha)^0|;$$

and for limit ordinals δ, $2(x,\delta)$ = supremum of $2(x,\alpha)$, $\alpha < \delta$.

If one assumes the Axiom of Choice this is the same as $2_{\alpha}^{|x|}$. Finally for each set x and $n \in \omega$, let $x^{(n)}$ be the set of all subsets of x having exactly n elements.

THEOREM. *Suppose κ is an infinite cardinal, I a set, $n \in \omega$, $\kappa > 2(I,n)^0$, and $\kappa^{(n+1)} = \cup C_i$ $(i \in I)$ is a partition of $\kappa^{(n+1)}$ into disjoint sets. Then there is a $y \in \kappa$ and an $i \in I$ such that $|y| > |I|$ and $y^{(n+1)} \subseteq C_i$.*

Proof. Notice first that since $\kappa^{(n+1)}$ can be well ordered there is no loss of generality in assuming that I can be well ordered or indeed that $I = |I|$. (It is here that we use the assumption that the C_i are disjoint.) The case $n = 0$ is now trivial. We proceed by induction. So suppose $n > 0$.

We shall define for each $\alpha \in \kappa$ a function f_α satisfying the following conditions:

(i) $\alpha \neq \beta$ implies $f_\alpha \neq f_\beta$;

(ii) the domain of f_α is an ordinal $\leqq \alpha$;

(iii) $\beta \in$ domain f_α implies there is a $\gamma < \alpha$ such that $f_\gamma = f_\alpha | \beta$. This γ is unique by (i) and will be denoted by $g(\alpha, \beta)$.

(iv) if $\beta \in$ domain f_α then $f_\alpha(\beta)$ is a function $t: (\beta + 1)^{(n)} \to I$ defined by $t(\{\tau_0, \cdots \tau_{n-1}\}) =$ that i such that $\{g(\alpha, \tau_0), \cdots, g(\alpha, \tau_{n-1}), \alpha\} \in C_i$.

From (iii) it follows that a necessary and sufficient condition that domain $f_\alpha = \beta$ is that $\beta \leqq$ domain f_α and for every $\beta < \alpha$, $f_\gamma | \beta \neq f_\alpha | \beta$. It follows that (i)–(iv) determines a unique set of functions $\{f_\alpha, \alpha \in k\}$.

Suppose β is an ordinal and $|\beta| \leqq 2(I, n-1)^0$. We assert that the set of $\alpha \in \kappa$ such that domain $f_\alpha = \beta$ has cardinality at most $2(I,n)$. To show this notice that for each $\gamma < \beta$ there is a canonical map of $\gamma^{(n)}$ into β^n (ordinary ordinal exponentiation). Thus f_α may be identified with a sequence of length β of sequences each of length β^n of elements from I. This in turn may be identified with a single sequence of length $\leqq \beta^{n+1}$ of elements of I. Since $2(I, n-1)^0 \geqq |I|$ the set of such sequences has cardinality $\leqq 2(I,n)$. Since there are at most $2(e,n)$ β's with $|\beta| \leqq 2(I, n-1)^0$ and $\kappa > 2(I,n)^0$, there must be some $\alpha \in \kappa$ with $|\text{domain } f_\alpha| > 2(I, n-1)^0$.

For this α, let $x = \{g(\alpha, \beta); \beta \in \text{domain } f_\alpha\}$. We define a partition of $x^{(n)}$ by letting $D_i = \{a \in x^{(n)}; a \cup \{\alpha\} \in C_i\}$. By the induction hypothesis there is a $y \subseteq x$ and an $i \in I$ such that $|y| > |I|$ and $y^{(n)} \leqq D_i$. Then $y^{(n+1)} \subseteq C_i$, for let $b \in y^{(n+1)}$, γ be the largest element of b; and $a = b - \{\gamma\}$. Then $a \subseteq \{a\} \in C_i$ and therefore by (iii) $b \in C_i$. The theorem is now proved.

Epilogue. Do our results apply to models of a Bernays type set theory? More generally, suppose $\mathfrak{A} = \langle A, V, E, R_1, R_2, \cdots \rangle$ is a relation system with V a unary, E a binary relation and $\langle V, E \rangle$ a model of ZF. Is it true that each of our theorems can be modified to say that there is an $\mathfrak{A}' = \langle A', V', E', R_1', R_2', \cdots \rangle$ such that $\mathfrak{A}' \succ \mathfrak{A}$ and $\langle V', E' \rangle$ is an extension of $\langle V, E \rangle$ of the required kind? An examination of the proofs shows that a sufficient condition for this to occur is that $\mathfrak{A}$ satisfy the axiom schema of replacement, namely:

$$(\forall x)[V(x) \rightarrow (\exists ! y)(V(y) \wedge \phi(x,y))] \rightarrow$$

$$(\forall z)[V(z) \rightarrow (\exists u)(\forall y)(y \in u \leftrightarrow V(y) \wedge (\exists x)[x \in z \wedge \phi(x,y)])]$$

for every formula ϕ.

In particular our results apply to Bernays-Morse set theory (see Kelley [9]). However they do not apply to Bernays–Gödel set theory (see [11]). Indeed, it is shown in [18] that there exists a countable model of Bernays-Gödel in which there is a formula defining a one-one correspondence between V and a subset of ω. Clearly this has no proper elementary extension in which ω is left fixed.

Do our results hold for models of Zermelo set theory (Z)? In general, the answer is yes except for those results which assert that the new model has ordinals which are greater than all ordinals of the old model. All the results of Section 2, Theorem 3.2, and Theorem 3.3 without part (iii), are still true if ZF is replaced by Z. Theorems 4.4 and 4.5 and its corollary are still true if $\mathfrak{A}$ is assumed to be a countable model of Z + AC in which all cardinals mentioned in the hypotheses exist. In each case the original proof still goes through. For models of Z the results may be stated in a more general form which applies to well ordered sets instead of initial ordinals. For instance, the conclusion of Theorem 2.2 holds if $\mathfrak{A}$ is a countable model of Z and in $\mathfrak{A}$, a is the set of all proper initial segments of a well-ordered structure whose order type is regular. And corresponding to Theorem 4.4 we have: If $\mathfrak{A}$ is a countable model of Z + AC and in $\mathfrak{A}$, $w = \omega$ and $b = R(\omega_1)$, then for every cardinal $k > \omega$ there exists $\mathfrak{B} \succ A$ such that w is fixed but b_F has power k.

Our results for models of Z can also be modified to apply to models $\mathfrak{A} = \langle A, V, E, R_1, R_2, \cdots \rangle$ such that $\langle V, E \rangle$ is a model of Z and $\mathfrak{A}$ satisfies the axiom scheme of subsets, namely:

$$(\forall z)[V(z) \rightarrow (\exists u)(\forall y)(y \in u \leftrightarrow y \in z \wedge \phi(y))]$$

for every formula ϕ.

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